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LETTER TO THE EDITOR

Extension of a theorem on super-multiplicative functions

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Abstract. This letter concerns a generalisation of the fundamental theorem on super-multiplicative functions for which the super-multiplicative inequality is replaced by $a_{n+m} \geqslant a_n a_{f(m)}$ with $\lim_{m\to\infty} m^{-1}f(m) = 1$.

A number of rigorous results concerning the limiting behaviour of the numbers of self-avoiding walks (Hammersley and Morton 1954), self-avoiding polygons (Hammersley 1961) and lattice animals (Klarner 1967) have been obtained by using properties of super-multiplicative and sub-multiplicative functions. The standard theorem on super-multiplicative functions (Hille 1948) is as follows: If a_n is a sequence of positive numbers such that $a_n^{1/n}$ is bounded above and

$$a_n a_m \leqslant a_{n+m}, \tag{1}$$

then there exists a positive constant μ such that

$$\sup_{n>0} n^{-1} \log a_n = \lim_{n\to\infty} n^{-1} \log a_n \equiv \log \mu < \infty. \tag{2}$$

(The corresponding theorem on sub-multiplicative functions can be obtained by setting $b_n = 1/a_n$ in the above statement.) Hammersley (1962) has considered a generalization of this theorem in which (1) is replaced by

$$a_{n+m} \geqslant g_{n+m} a_n a_m \tag{3}$$

and has examined the conditions on g_n for which $a_n^{1/n}$ tends to a finite limit. (An example which is of interest in terms of applications of these results is when g_n is a positive constant, $g_n = g$, for which the substitution $c_n = g a_n$ leads immediately to (2).)

A second generalisation which we consider here is when (1) is replaced by

$$a_n a_m \leqslant a_{n+f(m)}. \tag{4}$$

The question of interest is: for what class of functions $f(m)$ does $a_n^{1/n}$ tend to a finite limit? Our main result is the following theorem:

Theorem 1. Suppose a_n is a non-decreasing sequence of positive numbers such that $n^{-1} \log a_n$ is bounded above and $a_n a_m \leqslant a_{n+f(m)}$ for some positive function f which satisfies $\lim_{m\to\infty} m^{-1} f(m) = 1$. Then there is a positive constant μ such that the sequence $n^{-1} \log a_n$ converges to $\log \mu$ and the sequence a_n satisfies $a_n \leqslant \mu^{f(n)}$.

Proof. For fixed m, $p-1$ applications of the inequality (4) give

$$a_m^p \leqslant a_{m+(p-1)f(m)}, \tag{5}$$

which, coupled with the fact that a_n is non-decreasing, gives

$$a_m^p \leqslant a_n \tag{6}$$

for $n \geqslant m+(p-1)f(m)$. Taking logarithms and dividing by n gives

$$(mp/n)(m^{-1}\log a_m) \leqslant n^{-1}\log a_n.$$

This inequality is most effective when $m+(p-1)f(m) \leqslant n < m+pf(m)$ and, in this range,

$$mp/n > (m/f(m))(1-m/n). \tag{7}$$

Hence

$$\frac{\log a_n}{n} \geqslant \frac{m}{f(m)}\left(1-\frac{m}{n}\right)\frac{\log a_m}{m}. \tag{8}$$

Since $n^{-1}\log a_n$ is bounded above, we can define μ by

$$\log \mu = \limsup_{n\to\infty} n^{-1}\log a_n < \infty. \tag{9}$$

For a given $\epsilon > 0$ there exists an infinite set of numbers $S(\epsilon)$ such that

$$m^{-1}\log a_m \geqslant \log \mu - \epsilon/3, \qquad m \in S(\epsilon), \tag{10}$$

and we can choose a particular $m_0 \in S(\epsilon)$ with $m_0/f(m_0)$ sufficiently close to unity that

$$(m_0/f(m_0))(\log \mu - \epsilon/3) \geqslant \log \mu - \epsilon/2. \tag{11}$$

Now choose n_0 sufficiently large so that

$$(1-m_0/n)(\log \mu - \epsilon/2) \geqslant \log \mu - \epsilon \tag{12}$$

is satisfied for all $n \geqslant n_0 \geqslant m_0$. Then for $n \geqslant n_0(\epsilon)$

$$n^{-1}\log a_n \geqslant \log \mu - \epsilon, \tag{13}$$

and letting $\epsilon \to 0+$, (9) and (13) imply that

$$\lim_{n\to\infty} n^{-1}\log a_n = \log \mu < \infty. \tag{14}$$

Now that the existence of the limit is established, let $n \to \infty$ in (8), giving

$$a_m \leqslant \mu^{f(m)}, \qquad \forall m. \tag{15}$$

As one would anticipate, there is a corresponding theorem for the case in which the inequality is reversed which we state as:

Theorem 2. Suppose b_n is a non-decreasing sequence of positive numbers with the property that $b_n b_m \geqslant b_{n+f(m)}$ for some positive function f which satisfies $\lim_{m\to\infty} f(m)/m = 1$. Then there is a positive constant μ such that the sequence $n^{-1}\log b_n$ converges to $\log \mu$ and the sequence b_n satisfies $b_n \geqslant \mu^{f(n)}$.

Proof. The proof of this theorem follows the same lines as that of theorem 1, and we sketch only the minor differences involved. For fixed m, $p+1$ repeated applications of the inequality give

$$\frac{m(p+2)}{n}\frac{\log b_m}{m} \geqslant \frac{\log b_n}{n}, \tag{16}$$

and choosing p such that

$$m+pf(m)<n\leqslant m+(p+1)f(m) \tag{17}$$

leads to

$$\frac{m}{f(m)}\left(1+\frac{2f(m)-m}{n}\right)\frac{\log b_m}{m} \geqslant \frac{\log b_n}{n} \tag{18}$$

for $n>m$. Since the sequence b_n is non-decreasing and $b_n>0$, we can define

$$\lim_{n\to\infty}\inf n^{-1}\log b_n=\log\mu,$$

and the inequality (18) can then be used to show that, for n sufficiently large, $n^{-1}\log b_n \leqslant \log\mu+\epsilon$, for arbitrary $\epsilon>0$. Letting $n\to\infty$ in (18) then gives $b_m \geqslant \mu^{f(m)}$.

One immediate application of these theorems is to some problems in lattice statistics involving section graphs of the lattice, which will be discussed elsewhere. In other applications the conditions of the theorems may be satisfied for n or m sufficiently large. The conclusions of the theorems will still hold in these circumstances.

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